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LOCAL INSTABILITY OF PLATES WITH PRESSED-IN ANNULAR INCLUSIONS AT THE ELASTOPLASTIC BEHAVIOR OF MATERIALS

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The local instability of plates with annular inclusions is studied within the framework of exact three-dimensional equations. A numerical experiment is performed for the case where two rings are pressed in a plate made of the same material as the rings. The effect of the physicomechanical parameters of a medium on the critical contact pressures is studied.

It is well known that an analysis of the behavior of prestressed composite structures reduces to the formulation and solution of the problems of local instability [1, 2] at elastoplastic strains. In this paper, we study the buckling of a composite plate-like structure made of an elastoplastic material with translational hardening in an exact formulation based on the three-dimensional linearized theory of stability [3]. In this case, the loading function has the form [4]

$$F = (S_s^j - c_\beta(\varepsilon_s^j)^p)(S_s^j - c_\beta(\varepsilon_s^j)^p) - k_\beta^2 = 0, \tag{1}$$

and the relations of the associated flow rule are given by

$$(e_s^j)^p = \eta(S_s^j - c_\beta(\varepsilon_s^j)^p). \tag{2}$$

Here c_β are the hardening coefficients, k_β are the yield points, $S_s^j = \sigma_s^j - \sigma\delta_s^j$ is the deviatoric stress tensor, $\sigma = \sigma_k^k/3$, δ_s^j is the Kronecker symbol, ε_s^j are the strain-tensor components, e_s^j are the components of the strain-rate tensor, and η is a positive factor. The subscript s and the superscript j take on the values from 1 to 3. Summation is performed over repeated sub- and superscripts.

We consider the local instability of a plate-like structure which consists of an infinite plate with a circular hole of radius R_N into which a system of N rings pressed one in another is inserted with interference. The inner contour of the first ring is loaded by a uniformly distributed pressure q_0. It is assumed that the plate and the inclusions are made from different materials. On the contact lines of the units, the compressive forces $q_1, q_2, \ldots, q_N$ occur because of interferences. The quantities q_i $(i = 1, 2, \ldots, N)$ are such that the plastic regions completely cover the inner contours of the rings. We study the buckling of a plate-like structure within the framework of the second variant of the theory of small subcritical strains [5] with the use of the concept of continuous loading.

The subcritical stress–strain state of a plate-like structure under plane deformation is determined from the solution of two coupled problems of stress concentration. The stress–strain state of the ith ring is determined in the first problem, and the stress–strain state of the plate is determined in the second problem.

In the polar coordinates (r, θ), the subcritical strains and stresses of the ith ring have the forms [5]

$$(u_r)_i^p = \frac{a_2^i}{r}, \qquad (\varepsilon_r)_i^p = -(\varepsilon_\theta)_i^p = -\frac{k_i r^2 + 2\sqrt{2}\,a_2^i}{r^2\sqrt{2}(2 + c_i)},$$

$$(\sigma_r)_i^p = \frac{2c_i a_2^i}{2 + c_i}\Big(\frac{1}{R_{i-1}^2} - \frac{1}{r^2}\Big) + \frac{2\sqrt{2}\,k_i}{2 + c_i}\ln\frac{R_{i-1}}{r} - q_{i-1}, \tag{3}$$

$$(\sigma_\theta)_i^p = \frac{2c_i a_2^i}{2 + c_i}\Big(\frac{1}{R_{i-1}^2} + \frac{1}{r^2}\Big) + \frac{2\sqrt{2}\,k_i}{2 + c_i}\Big(\ln\frac{R_{i-1}}{r} - 1\Big) - q_{i-1}$$

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in the plastic region for $R_{i-1} < r < \psi_i$ and

$$(u_r)_i^e = b_1^i\Big(\frac{r}{3}+\frac{1}{r}\Big) - \frac{rq_i}{6}, \qquad (\sigma_r)_i^e = 2b_1^i\Big(1-\frac{1}{r^2}\Big) - q_i, \qquad (\sigma_\theta)_i^e = 2b_1^i\Big(1+\frac{1}{r^2}\Big) - q_i \tag{4}$$

in the elastic region for $\psi_i < r < R_i$. Here

$$b_1^i = -\frac{\psi_i^2(3\sqrt{2}\,k_i + c_i q_i)}{2(6 - c_i\psi_i^2)}, \qquad a_2^i = b_1^i\Big(1+\frac{\psi_i^2}{3}\Big) - \frac{q_i\psi_i^2}{6}.$$

The radius of the elastoplastic boundary ψ_i is determined from the equation

$$\frac{c_i}{2+c_i}\Big(\frac{1}{\psi_i^2}+\frac{1}{R_{i-1}^2}\Big)\frac{q_i\psi_i^2}{3} - \frac{\psi_i^2 R_i^2(3\sqrt{2}\,k_i + c_i k_i)}{6R_i^2 - c_i\psi_i^2}$$

$$\times\Big(\frac{1}{\psi_i^2}+\frac{1}{R_i^2} - \frac{c_i}{2+c_i}\Big(\frac{1}{\psi_i^2}+\frac{1}{R_{i-1}^2}\Big)\Big(\frac{\psi_i^2}{3R_i^2}+1\Big)\Big) + \frac{2\sqrt{2}\,k_i}{2+c_i}\Big(\ln\frac{\psi_i}{R_{i-1}}+1\Big) + q_{i-1} - q_i = 0. \tag{5}$$

In expressions (3)–(5), the subscript i takes on the values $1, 2, 3, \ldots, N$ (N is the number of rings), ψ_i and R_i are the radii of elastoplastic boundaries and the rings, respectively.

In polar coordinates, the subcritical strains and stresses of the plate have the forms

$$u_r^p = \frac{b_1}{r}, \quad \varepsilon_r^p = -\varepsilon_\theta^p = -\frac{2\sqrt{2}\,b_1 - kr^2}{r^2\sqrt{2}(2+c)}, \quad \sigma_r^p = \frac{2b_1 c}{2+c}\Big(1-\frac{1}{r^2}\Big) + \frac{2\sqrt{2}\,k}{2+c}\ln r - q_N,$$

$$\sigma_\theta^p = \frac{2b_1 c}{2+c}\Big(1+\frac{1}{r^2}\Big) + \frac{2\sqrt{2}\,k}{2+c}(\ln r + 1) - q_N \tag{6}$$

in the plastic region for $R_N < r < \gamma$ and

$$\sigma_r^e = -\sigma_\theta^e = -\frac{2b_1}{r^2} \tag{7}$$

in the elastic region for $\gamma < r < \infty$.

The radius of the elastoplastic boundary γ is determined from the equation

$$q_N - \frac{\sqrt{2}\,k}{2(2+c)}(4\ln\gamma + 2 + c\gamma^2) = 0. \tag{8}$$

In expressions (3), (4), and (6), u_r is the displacement-vector component and c and k are the hardening coefficient and yield point of the plate material, respectively.

Relations (3)–(8) are written in dimensionless form. The quantities that have the dimensions of stress and length are normalized to the shear modulus G and the outer radius of the first ring R_1, respectively. The superscripts e and p show that the quantities correspond to the elastic and plastic regions, respectively.

The stability analysis of the subcritical state (3)–(8) of a composite plate-like structure reduces to the solution of the variational equations of equilibrium for the plastic and elastic regions subject to corresponding boundary conditions. We write the equations of equilibrium [5]

$$\nabla_s(\sigma_j^s + \sigma_\alpha^{0s}\nabla^\alpha u_j) = 0, \tag{9}$$

the boundary conditions

$$N_s(\sigma_j^s + \sigma_\alpha^{0s}\nabla^\alpha u_j) = 0, \tag{10}$$

and the continuity conditions for stresses and displacements at the interfaces of the elastic and plastic regions

$$[N_s(\sigma_j^s + \sigma_\alpha^{0s}\nabla^\alpha u_j)]_\Sigma = 0, \qquad [u_j]_\Sigma = 0. \tag{11}$$

Here the square brackets denote the difference between the enclosed quantities in the elastic and plastic regions; Σ is the interface of these regions.

In the plastic and elastic regions of the incompressible elastoplastic model of a medium, a relation between the amplitudes of stresses and displacements has the form [5]

$$\sigma_j^s = (a_{s\alpha}g^{\alpha\alpha}\nabla_\alpha u_\alpha + \rho)g_j^s + (1 - g_j^s)g^{ss}G_j^s(\nabla_s u_j + \nabla_j u_s), \tag{12}$$

where ρ is the Lagrange multiplier and g_j^s are the components of the metric tensor (no summation over the subscripts s and j is performed). The quantities $a_{s\alpha}$ and G_j^s can be written in the form

$$a_{s\alpha} = 2\mu g_{s\alpha} - \frac{4\mu^2\chi f^0_{ss} f^0_{\alpha\alpha}}{k^2_\beta(2\mu + c_\beta)}, \qquad G^s_j = \mu = G, \qquad f^0_{sj} = S^0_{sj} - c_\beta \varepsilon^{0p}_{sj}. \tag{13}$$

The value of $\chi = 1$ corresponds to an elastoplastic medium [4], and $\chi = 0$ to an elastic medium. The superscript 0 denotes the subcritical state.

With allowance for the incompressibility condition, Eqs. (9)–(13) constitute a coupled boundary-value static-stability problem for the amplitudes of the displacement-vector components u, v, and w and the hydrostatic pressure p in the elastic and plastic regions of the rings and the plate. The nontrivial solution of this problem corresponds to the instability of the subcritical state. To find eigenvalues of the problem, we approximate the displacements and the hydrostatic pressure in the elastic and plastic regions of the rings and the plate by double trigonometric series [5]:

$$\begin{gathered} u = \sum_n^\infty \sum_m^\infty A_{nm}(r)\cos(m\theta)\cos(nz), \qquad v = \sum_n^\infty \sum_m^\infty B_{nm}(r)\sin(m\theta)\cos(nz), \\ w = \sum_n^\infty \sum_m^\infty C_{nm}(r)\cos(m\theta)\sin(nz), \qquad p = \sum_n^\infty \sum_m^\infty D_{nm}(r)\cos(m\theta)\cos(nz). \end{gathered} \tag{14}$$

Here n and m are the wavenumbers.

Substituting the functions u, v, w, and p into the linearized stability equations (9) and taking into account relations (12) and (13) and the incompressibility condition, after some manipulations we obtain the following infinite system of differential equations for A_{nm} and B_{nm}:

$$\begin{gathered} \xi_1 A(r) + \xi_2 A'(r) + \xi_3 A''(r) + \xi_4 B(r) + \xi_5 B'(r) + \xi_6 B''(r) + \xi_7 B'''(r) = 0, \\ \xi_8 A(r) + \xi_9 A'(r) + \xi_{10} A''(r) + \xi_{11} A'''(r) + \xi_{12} B(r) + \xi_{13} B'(r) + \xi_{14} B''(r) = 0. \end{gathered} \tag{15}$$

Here

$$\xi_1 = \sigma^0_\theta(1 - m^2) - m^2 - 2r\sigma^0_{\theta,r} - n^2r^2 + 1, \qquad \xi_2 = r^2\sigma^0_{r,r} - r(2\sigma^0_\theta + 1 - \sigma^0_r),$$

$$\xi_3 = r^2(1 - 2a_0 + \sigma^0_r), \qquad \xi_4 = (1 + \sigma^0_\theta)\Big(\frac{1}{m} - m\Big) - \frac{r\sigma^0_{\theta,r}(1 + m^2)}{m^3} - \frac{r^2n^2}{m^3},$$

$$\xi_5 = r\Big[(2a_0 - 1 - \sigma^0_\theta)m - \frac{1}{m}(1 + \sigma^0_\theta)\Big] + \frac{r^3}{m}(\sigma^0_{r,rr} - n^2) + \frac{r^2}{m} 2\sigma^0_{r,r},$$

$$\xi_6 = \frac{2r^2}{m}(\sigma^0_r + \sigma^0_{r,r} + 1), \qquad \xi_7 = \frac{r^3}{m}(\sigma^0_r + 1),$$

$$\xi_8 = nr^2(a_0 - 1 - 2\sigma^0_\theta) + \frac{m^2r}{n}(1 + \sigma^0_\theta) - \frac{r}{n}(1 + \sigma^0_r - r\sigma^0_{r,r}),$$

$$\xi_9 = nr^3(1 - a_0) - \frac{m^2r}{n}(1 + \sigma^0_\theta) - \frac{r}{n}(1 + \sigma^0_r - r\sigma^0_{r,r}), \quad \xi_{10} = \frac{r^2}{n}(2 + 2\sigma^0_r + r\sigma^0_{r,r}),$$

$$\xi_{11} = \frac{r^3}{n}(1 + \sigma^0_r), \quad \xi_{12} = mnr^2a_0 - \frac{n^3r^2}{m} - \Big(\frac{1}{m} + \frac{m^3}{nr^2}\Big)(\sigma^0_\theta + 1) + \frac{m}{nr^2}(1 + \sigma^0_r - r\sigma^0_{r,r}),$$

$$\xi_{13} = \frac{rm}{n}(\sigma^0_{r,r} - \sigma^0_r - 1) - \frac{r^3n}{m}(1 + \sigma^0_r + r\sigma^0_{r,r}), \quad \xi_{14} = (1 + \sigma^0_r)\Big(\frac{mr^2}{n} + \frac{r^4n}{m}\Big).$$

In (15) and below, the subscript nm at A and B is dropped.

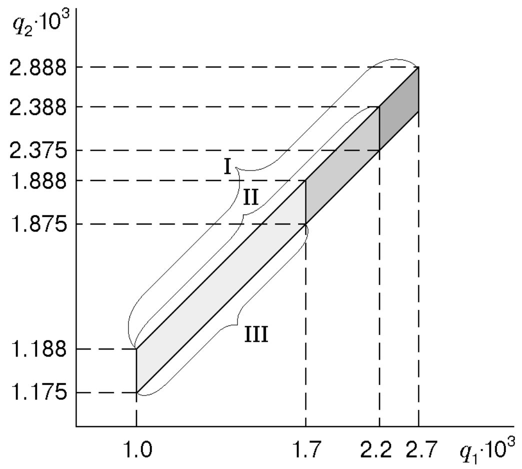

Fig. 1

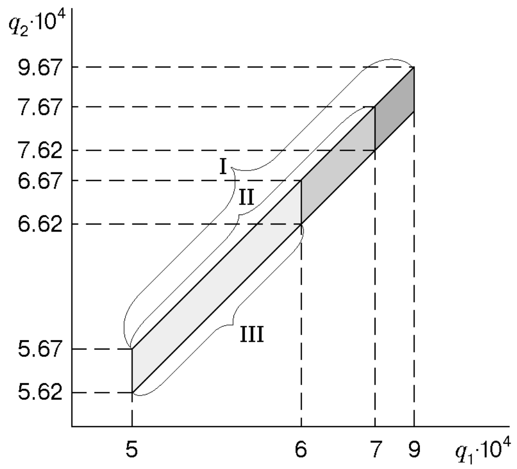

Fig. 2

Using relations (11) and (13), we write the continuity conditions for displacement perturbations

$$A_1^p \Big\{ \frac{2}{R_0} \left[a_0 - (\sigma_\theta^0)_1^p - 1 \right] \Big\} + (A')_1^p [2 - 2a_0 + (\sigma_r^0)_1^p]$$

$$+ B_1^p \Big\{ \frac{1}{R_0} \Big[m a_0 - \frac{1}{m} (1 + (\sigma_\theta^0)_1^p) + m^2 ((\sigma_\theta^0)_1^p + 1 - a_0) \Big] - \frac{n^2 R_0}{m} \Big\}$$

$$+ (B')_1^p \frac{1}{m} [1 + (\sigma_r^0)_1^p + R_0 (\sigma_{\theta,r}^0)_1^p] + (B'')_1^p \frac{R_0}{m} [1 + (\sigma_r^0)_1^p] = 0, \tag{16}$$

$$m A_1^p + B_1^p - R_0 (B')_1^p [1 + (\sigma_r^0)_1^p] = 0,$$

$$A_1^p \Big[n - \frac{1 + (\sigma_r^0)_1^p}{n R_0^2} \Big] + \frac{1 + (\sigma_r^0)_1^0}{n R_0} \Big[(A')_1^p - \frac{m}{R_0} B_1^p + m (B')_1^p + R_0 (A'')_1^p \Big] = 0$$

at the inner contour of the first ring ($r = R_0$) and

$$A_i^p \frac{1}{\psi_i} - (A')_i^p + B_i^p \frac{m}{\psi_i} + (B')_i^p \Big\{ \frac{\psi_i}{2 m a_0} [(\sigma_{r,r}^0)_i^p - (\sigma_{r,r}^0)_i^e] \Big\} + \frac{\psi_i}{2 m a_0} [(\sigma_r^0)_i^e + 1][(B'')_i^p - (B'')_i^e] = 0,$$

$$(A'')_i^p - (A'')_i^e = 0 \tag{17}$$

at the interface of the elastic and plastic regions of the inclusions ψ_i ($i = 1, 2, \ldots, N$). One can obtain conditions similar to (17) at the interface of the elastic and plastic regions of the plate γ.

Using (10), we obtain the relations

$$A_{i+1}^p - (A')_{i+1}^p + \frac{m}{R_i} B_{i+1}^p + (B')_{i+1}^p \Big\{ \frac{R_i}{2 m a_0} [(\sigma_{r,r}^0)_{i+1}^p - (\sigma_{r,r}^0)_i^e] \Big\} + \frac{R_i}{2 m a_0} [(\sigma_r^0)_i^e + 1][(B'')_{i+1}^p - (B'')_i^e] = 0, \tag{18}$$

$$(A'')_{i+1}^p - (A'')_i^e = 0 \qquad (i = 1, 2, \ldots, N)$$

which must be satisfied at the interface of the ith and $(i+1)$th rings. The condition of local perturbations $u_j \to 0$ as $r \to \infty$ ($j = 1$, 2, 3) yields

$$(A')^e = 0, \qquad (A'')^e = 0, \qquad (B')^e = 0, \qquad (B'')^e = 0. \tag{19}$$

Since we failed to find an analytical solution of the boundary-value problem (15)–(19), we seek its approximate solution by the finite-difference method [6]. As a result, we obtain an infinite system of homogeneous algebraic equations linear in the parameters A_{nm} and B_{nm} that must be solved with allowance for Eqs. (5) and (8) determining the elastoplastic boundaries of the rings and the plate, respectively. Local buckling occurs for the nonzero values of A_{nm} and B_{nm}. The parameter q_i is minimized with respect to the step of the finite-difference grid h, the wavenumbers m and n, and the material and structure parameter λ_j.

As a result, we arrive at the problem of multidimensional optimization of the quantities q_i $(i = 1, 2, \ldots, N)$ with respect to m and n. This problem reduces to the determination of a condition under which the determinant of the resulting algebraic system vanishes: $\det(q_i, m, n, \lambda_j) = 0$.

As an example, we consider the case where two rings ($N = 2$) are pressed in a plate. The material of the plate and the rings is assumed to be the same. As a material, we used St. 3 steel ($c = 0.002$, $k = 0.0014$, and $G = 0.81 \cdot 10^6$ kg/cm^2) and copper ($c = 0.006$, $k = 0.0005$, and $G = 0.81 \cdot 10^6$ kg/cm^2).

Figure 1 shows regions of the critical contact pressures q_1 and q_2 for $n = m = 3$ and $q_0 = 0$ at the inner contour of the first steel ring. Region I corresponds to $R_0 = 0.002$ and $R_2 = 1.1$ and comprises region II ($R_0 = 0.007$ and $R_2 = 1.1$), which, in turn, comprises region III ($R_0 = 0.02$ and $R_2 = 1.1$).

Figure 2 shows regions of the critical contact pressures q_1 and q_2 for $n = m = 4$ and $q_0 = 0$ at the inner contour of the first copper ring. Region I corresponds to $R_0 = 0.002$ and $R_2 = 1.1$ and comprises region II ($R_0 = 0.007$ and $R_2 = 1.1$), which, in turn, comprises region III ($R_0 = 0.02$ and $R_2 = 1.1$).

An analysis of the numerical results shows the following:

— the region of the critical parameters q_1 and q_2 increases with the width of the internal ring;

— the region of the critical parameters q_1 and q_2 decreases with the physicomechanical characteristics c and k;

— a one-term approximation of the displacements gives the overestimated critical parameters q_1 and q_2.

For a St. 3 steel plate with inclusions, the local-buckling mode is nonsymmetric: three half-waves form in the direction of the θ axis and three half-waves form in the z direction. For copper, local buckling occurs for $m = n = 4$. Setting $R_2 = R_1$, we obtain the critical load q_1 of a plate with one annular inclusion.

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